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THE NECESSARY CONDITIONS FOR THE STABILITY OF STEADY MOTIONS OF SYSTEMS WITH CONSTRAINTS PRODUCED BY LARGE POTENTIAL FORCES†

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The problem of the necessary conditions for the stability of steady motions for systems with constraints produced by large potential forces is investigated. An augmented matrix, similar to that arising under conditions when the restriction of quadratic forms to a linear manifold is sign-definite, is introduced to investigate the eigenvalues of the linearized system. The question of the correctness of the realization of unilateral constraints in special cases of zero reaction is discussed using an example.

As is well known, Routh's method [1, 2] and its modifications (see, for example, [3–5]) enable one not only to solve effectively the problem of the existence of steady motions of mechanical systems, subject to constraints or possessing first integrals, but also to investigate the sufficient conditions for their stability or instability. At the same time, for systems subject to unilateral constraints, investigations of the necessary conditions for stability are made difficult by certain complexities related to the construction of the equations of perturbed motion. Below we discuss one of the possible ways of overcoming these difficulties in the case when the mechanical nature of the constraint is known.

1. THE EQUATIONS OF MOTION OF A MECHANICAL SYSTEM CONSTRAINED BY A LARGE POTENTIAL FORCE

Consider the motion of a mechanical system, for which the expressions for the kinetic and potential energy have the form

$$T = T(\mathbf{x}, \dot{\mathbf{x}}), \quad U = U(\mathbf{x}), \quad \mathbf{x} \in R^n \tag{1.1}$$

We will assume that an additional force with a potential (compared with [6, 5])

$$U_N = \frac{1}{2} N \varphi(\mathbf{x}) \tag{1.2}$$

which depends on the positive parameter N, also acts on the system, where

$$\varphi(\mathbf{x}) = f^2(\mathbf{x}) \tag{1.3}$$

or

$$\varphi(\mathbf{x}) = \begin{cases} f^2(\mathbf{x}), & \mathbf{x} \in \mathscr{C}_+ \cup \mathscr{C} \\ 0, & \mathbf{x} \in \mathscr{C}_- \end{cases} \tag{1.4}$$

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The regions $\mathscr{E}_\pm$ and the surface $\mathscr{E}$ are defined by the relations

$$\mathscr{E}_- = \{\mathbf{x}:\ f(\mathbf{x})<0\},\quad \mathscr{E}_+ = \{\mathbf{x}:\ f(\mathbf{x})>0\},\quad \mathscr{E} = \{\mathbf{x}:\ f(\mathbf{x}) = 0\} \tag{1.5}$$

respectively. We will assume that $f(\mathbf{x})$ is a continuous function in R^n, and the smooth surface $\mathscr{E}$ can be split into n-dimensional regions $\mathscr{E}_-$ and $\mathscr{E}_+$.

We will assume that, for sufficiently large values of the parameter N, a force with the potential (1.2), (1.3) produces the bilateral constraint

$$f(\mathbf{x}) = 0,\quad \mathbf{x}\in R^n \tag{1.6}$$

while a force with potential (1.2), (1.4) produces the unilateral constraint

$$f(\mathbf{x})\le 0,\quad \mathbf{x}\in R^n \tag{1.7}$$

Since all the functions introduced above are continuous, the equations of motion can be represented in the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{x}}} = \frac{\partial L}{\partial \mathbf{x}} - \frac{\partial U_N}{\partial \mathbf{x}},\quad L = T - U \tag{1.8}$$

2. STEADY MOTIONS

According to Routh's theory [1–5], the steady motion of the system considered can be obtained as critical points of the potential

$$W(\mathbf{x}; N) = U(\mathbf{x}) + U_N(\mathbf{x}) \tag{2.1}$$

They are defined by the relations

$$W_{\mathbf{x}}(\mathbf{x}; N) = U_{\mathbf{x}}(\mathbf{x}) + Nf(\mathbf{x})f_{\mathbf{x}}(\mathbf{x}) = 0 \tag{2.2}$$

which hold both for the potential (1.2), (1.3), and for the potential (1.2), (1.4) in the region $\mathscr{E} \cup \mathscr{E}_+$. For potential (1.2), (1.4) in the region $\mathscr{E}_-$, the equations of steady motion have the form

$$W_{\mathbf{x}}(\mathbf{x}; N) = U_{\mathbf{x}}(\mathbf{x}) = 0$$

We will introduce the variable

$$\lambda = Nf(\mathbf{x})$$

as was done previously in [5]. Then, system (2.2), consisting of n equation in n variables, becomes the equivalent system

$$U_{\mathbf{x}} + \lambda f_{\mathbf{x}} = 0,\quad f(\mathbf{x}) = \varepsilon\lambda,\quad \varepsilon = N^{-1} \tag{2.3}$$

consisting of $n + 1$ equations in $n + 1$ unknowns. As $\varepsilon \to 0$ the second equation is converted into Eq. (1.5) of the surface $\mathscr{E}$.

The solution of system (2.3), presented in the form of a formal series in the parameter ε, has the form

$$\mathbf{x} = \mathbf{x}_0 + \varepsilon\mathbf{x}_1 + \dots,\quad \lambda = \lambda_0 + \varepsilon\lambda_1 + \dots \tag{2.4}$$

where the quantities x_0 and λ_0 satisfy the system

$$U_{\mathbf{x}}^0 + \lambda_0 f_{\mathbf{x}}^0 = 0,\quad f^0 = 0 \tag{2.5}$$

$$f^0 = f(\mathbf{x}_0),\quad f_{\mathbf{x}}^0 = f_{\mathbf{x}}(\mathbf{x}_0),\quad U_{\mathbf{x}}^0 = U_{\mathbf{x}}(\mathbf{x}_0)$$

If the matrix

$$\mathbf{A} = U_{\mathbf{xx}}(\mathbf{x}_0) + \lambda_0 f_{\mathbf{xx}}(\mathbf{x}_0) \tag{2.6}$$

is non-degenerate, the quantities $\mathbf{x}_1$ and λ_1 can be represented in the form

$$\mathbf{x}_1 = \lambda_0(\mathbf{A}^{-1} f_{\mathbf{x}}^0, f_{\mathbf{x}}^0)^{-1} \mathbf{A}^{-1} f_{\mathbf{x}}^0, \quad \lambda_1 = -\lambda_0(\mathbf{A}^{-1} f_{\mathbf{x}}^0, f_{\mathbf{x}}^0)^{-1}$$

in the case when the constraint is stressed. Since

$$f(\mathbf{x}_0 + \varepsilon \mathbf{x}_1 + \ldots) = f^0 + \varepsilon(f_{\mathbf{x}}^0, \mathbf{x}_1) + \ldots = \varepsilon\lambda_0 + \ldots$$

and $\varepsilon > 0$ when $\lambda_0 > 0$ the critical point is situated inside the region $\mathscr{E}_+$, and the constraint is stressed irrespective of whether it is unilateral or bilateral. When $\lambda_0 < 0$ the critical point, which satisfies relations (2.5), is situated inside the region $\mathscr{E}_-$, which it cannot be when realizing a unilateral constraint with the assumptions made above. Finally, the case $\lambda_0 = 0$ requires the consideration of higher approximations.

3. THE NECESSARY CONDITIONS FOR STABILITY

The sufficient conditions for stability of the steady motions obtained within the framework of Routh's theory as $N \to \infty$, i.e. as $\varepsilon \to 0$, were established earlier in [5]. We will investigate the necessary conditions for stability. First of all, using the notation introduced above, we will represent the equations of motion in the form

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{x}}} = \frac{\partial L}{\partial \mathbf{x}} - \lambda\frac{\partial f}{\partial \mathbf{x}}, \quad f(\mathbf{x}) = \lambda\varepsilon \tag{3.1}$$

These equations hold everywhere if the constraint is bilateral, and in the region $\mathscr{E} \cup \mathscr{E}_+$ if the constraint is unilateral. We will consider the case when any of these conditions is satisfied. Then, the equations of motion (3.1), linearized in a small neighbourhood of the steady motion obtained, have the form

$$\frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j}\delta\ddot{x}_j + \frac{\partial^2 L}{\partial \dot{x}_i \partial x_j}\delta\dot{x}_j = \frac{\partial^2 L}{\partial x_i \partial \dot{x}_j}\delta\dot{x}_j + \frac{\partial^2 L}{\partial x_i \partial x_j}\delta x_j - \frac{\partial f}{\partial x_i}\delta\lambda - \lambda\frac{\partial^2 f}{\partial x_i \partial x_j}\delta x_j \tag{3.2}$$

$$\frac{\partial f}{\partial x_j}\delta x_j = \varepsilon\delta\lambda \tag{3.3}$$

All the partial derivatives in Eqs (3.2) and (3.3) are calculated at the critical point ($\mathbf{x} = \mathbf{x}_\varepsilon$, $\dot{\mathbf{x}} = 0$, $\lambda = \lambda_\varepsilon$). By substituting expansions (2.4) into Eqs (3.2) and (3.3), in the limit as $\varepsilon \to 0$ we can represent these relations in the form

$$\mathbf{P}\delta\ddot{\mathbf{x}} + \mathbf{G}\delta\dot{\mathbf{x}} + \mathbf{A}\delta\mathbf{x} + \mathbf{F}\delta\lambda = 0, \quad (\mathbf{F}, \delta\mathbf{x}) = 0 \tag{3.4}$$

(compared with [7, Chapter 5, p. 38]). Then, the characteristic equation can be represented in the form

$$\det\mathbf{J}(\mu) = 0, \quad \mathbf{J}(\mu) = \begin{Vmatrix} \mathbf{H}(\mu) & \mathbf{F} \\ \mathbf{F}^T & 0 \end{Vmatrix}, \quad \mathbf{H}(\mu) = \mathbf{P}\mu^2 + \mathbf{G}\mu + \mathbf{A} \tag{3.5}$$

The matrix $\mathbf{J}(0)$ is identical with the augmented matrix of the second variation of the potential, known from [4] for systems subject to a bilateral constraint, and from [5] for systems subject to a unilateral constraint.

In fact, substituting the relations

$$\delta\mathbf{x} = e^{\mu t}\delta\mathbf{x}_0, \quad \delta\lambda = e^{\mu t}\delta\lambda_0 \tag{3.6}$$

into Eq. (3.4), we obtain

$$\mathbf{H}(\mu)\delta\mathbf{x}_0 + \mathbf{F}\delta\lambda_0 = 0, \quad (\mathbf{F}, \delta\mathbf{x}_0) = 0 \tag{3.7}$$

whence the assertion also follows. Moreover, since

$$\delta \mathbf{x}_0 = -\mathbf{H}^{-1}(\mu)\mathbf{F}\delta\lambda_0$$

we have from the second relation of (3.7)

$$(\mathbf{F}, \mathbf{H}^{-1}(\mu)\mathbf{F}) = 0 \tag{3.8}$$

In the simplest situation, when there are only two degrees of freedom, the assertion in [7] on possible stabilization of a system when there is a new constraint immediately follows from relation (3.8).

Remark. Expanding determinant (3.5) with respect to the last column, we obtain a polynomial of degree $2(n-1)$. This indicates that the use of the augmented matrix enables one, in advance, to eliminate the pair of zero roots, obtained in the direct calculation of the determinant of the unaugmented matrix.

4. THE STRUCTURE OF THE CHARACTERISTIC POLYNOMIAL OF A SYSTEM WITH THREE DEGREES OF FREEDOM, ON WHICH A SINGLE CONSTRAINT IS IMPOSED

Suppose

$$\mathbf{P} = \mathbf{I}, \quad G = \begin{Vmatrix} 0 & g_3 & -g_2 \\ -g_3 & 0 & g_1 \\ g_2 & -g_1 & 0 \end{Vmatrix}, \quad \mathbf{A} = \operatorname{diag}(a_1, a_2, a_3), \quad \mathbf{F} = \operatorname{col}(f_1, f_2, f_3)$$

The characteristic equation is then given the relation

$$\begin{vmatrix} \mu^2 + a_1 & g_3\mu & -g_2\mu & f_1 \\ -g_3\mu & \mu^2 + a_2 & g_1\mu & f_2 \\ g_2\mu & -g_1\mu & \mu^2 + a_3 & f_3 \\ f_1 & f_2 & f_3 & 0 \end{vmatrix} = 0 \tag{4.1}$$

The function on the left-hand side of (4.1) is even in μ. Consequently, assuming $\nu = \mu^2$ and changing the sign, we obtain

$$a\nu^2 + b\nu + c = 0$$
$$a = \sum_{(1,2,3)} f_1^2, \quad b = \sum_{(1,2,3)} f_1^2(a_2 + a_3) + \left[\sum_{(1,2,3)} f_1 g_1\right]^2, \quad c = \sum_{(1,2,3)} f_1^2 a_2 a_3 \tag{4.2}$$

Moreover, since the steady motion obtained is assumed to be non-special, we can assume that

$$a = f_1^2 + f_2^2 + f_3^2 = 1 \tag{4.3}$$

The necessary conditions for stability turn out to be satisfied if both roots of Eq. (4.2) are real and negative. By virtue of the last assumption, these conditions have the form

$$b^2 - c > 0, \quad b > 0, \quad c > 0 \tag{4.4}$$

Remark. Generally speaking, the problem considered looks simpler than the problem of the existence and stability of periodic motions of systems subject to unilateral constraints. As regards publications on this optic, going back, probably, to [8–11], there are detailed reviews in [12–15].

5. THE GENERALIZED THREE-BODY PROBLEM

In order to illustrate the results obtained, we will consider the plane problem of the motion of two tethered massive points, which interact with a third massive point by forces of Newtonian attraction. This problem was considered in [15] (see also [16]) on the assumption that the points are connected by inextensible weightless rods and form a dumbell; it was shown, in particular, that there are regions in parameter space such that, for appropriate values of these parameters, gyroscopic stabilization of "triangular" steady motions is possible.

As an extension of these results, the conditions for the existence and stability of steady motions were considered for a broad class of possible interaction potentials between the points forming the dumbell [17].

In order to apply the results obtained to the problem described above, it is sufficient to investigate the sign of the force which produces the corresponding constraint. If r_1 and r_2 are the distances from the point m_0 to the points m_1 and m_2 forming the dumbell, respectively, and α is the angle between m_0m_1 and m_0m_2, the reduced potential has the form [17]

$$W = \frac{p^2}{2J} + \Pi + E(l), \quad l = (r_1^2 + r_2^2 - 2r_1r_2\cos\alpha)^{1/2} \tag{5.1}$$

where $\Pi = -\gamma m_0(m_1/r_1 + m_2/r_2)$ is the Newtonian potential, E is the interaction potential between the points m_1 and m_2, and p is the constant of the cyclic integral. The function

$$J = \frac{1}{m}[m_1(m_0 + m_2)r_1^2 + m_2(m_0 + m_1)r_2^2 - 2m_1m_2r_1r_2\cos\alpha], \quad m = m_0 + m_1 + m_2$$

describes the moment of inertia of the whole system about its centre of mass.

The equations of the critical points of the reduced potential (5.1), i.e. the equations of the steady configurations, have the form [17]

$$\begin{aligned}\frac{\partial W}{\partial r_1} &= -\frac{p^2}{mJ^2}[m_1(m_0 + m_2)r_1 - m_1m_2r_2\cos\alpha] + \gamma\frac{m_0m_1}{r_1^2} + \\ &+ \frac{r_1 - r_2\cos\alpha}{l}E'(l) = 0 \quad (1 \leftrightarrow 2)\end{aligned} \tag{5.2}$$

$$\frac{\partial W}{\partial \alpha} = \left[-\frac{p^2}{mJ^2}m_1m_2\sin\alpha + \frac{E'(l)}{l}\right]r_1r_2\sin\alpha = 0 \tag{5.3}$$

According to Eqs (5.3), there is a class of "triangular motions" such that

$$E'(l) = \frac{p^2 l}{mJ^2}m_1m_2\sin\alpha \tag{5.4}$$

We will assume that the tensile strength of the tether is given by the potential

$$E(l) = \begin{cases} N(l - l_0)^2/2, & l - l_0 \geq 0 \\ 0, & l - l_0 < 0 \end{cases} \tag{5.5}$$

where l_0 is the length of the tether in the unstretched state. Then, by virtue of expressions (5.4) and (5.5) in the class of triangular motions the tether stretches, and the results related to the existence and stability of steady motions, obtained previously in [15, 16] for a constraint realized using a rod, remain true for a constraint realized using a tether.

6. THE REALIZATION OF UNILATERAL CONSTRAINTS IN SPECIAL CASES

The use of large potential forces to model unilateral constraints in special cases requires certain precautions. As an example, we will consider the motion of a heavy bead, freely sliding along a vertical

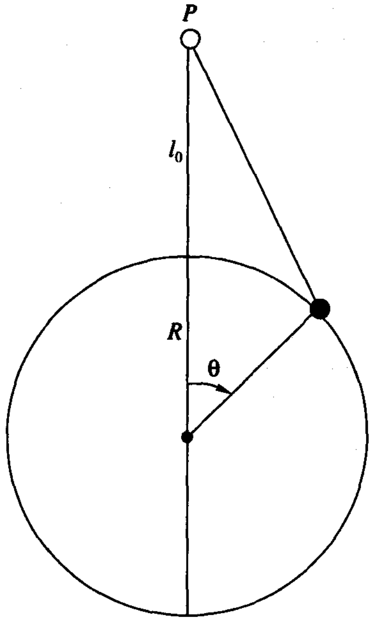

Fig. 1

collar in the form of a circle of radius R. Suppose θ is the angle of deflection of the bead from the ascending vertical (see the figure), the bead is connected by a massless tether to the point P, situated above the vertical diameter of the circle at a distance $\rho = R + l_0$ from its centre, and l_0 is the length of the tether in the unstretched state.

We will obtain the equilibrium positions of the system and investigate their stability in the case when the potential of the elastic forces has the form

$$U_E = \frac{N}{2}\varphi(l), \quad \varphi(l) = \begin{cases} 0, & l < l_0 \\ (l - l_0)^2, & l \geq l_0 \end{cases}$$

where l is the length of the stretched tether. By the cosine theorem

$$l = [\rho^2 - 2\rho R\cos\theta + R^2]^{1/2}, \quad \partial l/\partial \; = \rho R\sin\theta/l$$

Since the potential energy of a uniform gravitational field is equal to $U_N = mgR\cos\theta$, the equilibria are found from the equation

$$\partial U/\partial\theta = \varphi(\theta)\sin\theta = 0, \quad U = U_N + U_E, \quad \varphi(\theta) = -mgR + N(l - l_0)\rho R/l$$

The system has four different equilibria. Where $\theta = 0$ we have the upper equilibrium and when $\theta = \pi$ we have the lower equilibrium. There are also two "oblique" symmetrical equilibria, for which the length of the tether is given by

$$l = Nl_0\rho/(N\rho - mg)$$

The oblique equilibria approach the upper equilibrium as $N \to \infty$. They exist when $N > N^*$, where

$$N^* = mg(R + \rho)/(2R\rho)$$

When $N = N^*$ the oblique equilibria coincide with the lower equilibrium, and when $N < N^*$ the tether is tensionless.

In order to investigate the stability of the equilibria we will calculate the second derivative of the potential. It has the form

$$\partial^2 U/\partial\theta^2 = \varphi'(\theta)\sin\theta + \varphi(\theta)\cos\theta$$

At the upper and lower equilibria the first term on the right-hand side vanishes. At the oblique equilibria the second term vanishes. Moreover, at the upper equilibrium

$$\partial^2 U(0)/\partial\theta^2 = -mgR$$

and it is unstable for all values of the constant of elasticity N. At the lower equilibrium

$$\partial^2 U(\pi)/\partial\theta^2 = 2(N - N^*)R^2\rho/(R + \rho)$$

It is unstable when $N > N^*$ and stable when $N < N^*$. An investigation of the intermediate situation requires a calculation of higher derivatives.

Hence, the results obtained are somewhat depressing. In fact, the only kinematically possible position of the system with a tensionless tether turns out to be unstable, i.e. it is physically unrealizable for any value of the constant of elasticity as large as desired. In this case, the oblique equilibria turn out to be stable and physically realizable, but disappear when the constant of elasticity has arbitrarily large values.

The example considered shows that it makes sense to introduce the idea of the mobility of the system, based on an energy criterion. We will take as the basis of this, the idea of regions of possible motion (see, for example, [18]).

Definition. Suppose a mechanical system, performing motion under the action of active forces with a potential $U(\mathbf{x})$ and under the action of forces with a potential $U_N(\mathbf{x})$, which produce a constraint in the limit as $N \to \infty$, allow of a first integral (the Painlevé–Jacobi energy integral)

$$\mathcal{J} = T(\dot{\mathbf{x}}, \mathbf{x}) + U(\mathbf{x}) + U_N(\mathbf{x}) = h$$

where $T \geq 0$, and the condition $T = 0$ implies that the equality $\mathbf{x} = 0$ is satisfied. The region of configurational space $\mathcal{A}_{N,h} = \{\mathbf{x}: U + U_N \leq h\}$ will be called the region of mobility of the system at the level h of the Painlevé–Jacobi energy integral. The region of configuration space to which the region of mobility approaches as $N \to \infty$, will be called the limit region of mobility at the level h of the integral $\mathcal{J}$.

In the example considered above, for values of $h < mgR$ the region of mobility turns out to be empty for fairly large values of N. For these values of N the motion turns out to be impossible for the values of the energy integral indicated. In this case the limit region of mobility also turns out to be empty. However, when $h \geq mgR$ the region of mobility is not empty for all values of N, and the motion turns out to be possible. In this case the limit region of mobility consists of exactly one point, as was shown above.

It is clear that the structure of the region of the mobility in general depends considerably on the structure of the potential of the active forces.

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